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Derivatives of the Algebraic Polynomials of Best Approximation

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I. INTRODUCTION

Let $C[a, b]$ be the space of continuous real valued functions defined on the compact interval $[a, b]$, endowed with the supremum norm denoted by $\| \|$. Let P_n be the algebraic polynomial of degree at most n of best approximation to $f \in C[a, b]$. The main purpose of this paper is the investigation of the behavior, as $n \to \infty$, of $\| P_n^{(k)} \|$ and $\| P_n^{(k)} \|_{[\alpha,\beta]} = \max_{\alpha \leqslant x \leqslant \beta} | P_n^{(k)}(x)|$, $a < \alpha < \beta < b$. In a subsequent work we shall apply our results to the problem of lacunary approximation.

In this paper, P_n, Q_n, R_n will always denote algebraic polynomials of degree at most n. The sentence: "Let P_n be the polynomial of best approximation to $f \in C[a, b]$" means that P_n is the polynomial of best approximation to f on $[a, b]$. All constants appearing in this paper depend on a and b.

We now state the theorems on which our study relies. Let $f \in C^N[a, b]$, the subspace of $C[a, b]$ of N-times continuously differentiable functions; let $E_n(f) = \| P_n - f \|$.

THEOREM 1.1 (Jackson [7, p. 127]). *There exists a constant K, which depends on N, such that*

$$E_n(f) \leqslant \frac{K}{n^N} \omega \left(f^{(N)}, \frac{1}{n}\right), \qquad n \geqslant 1,$$

where $\omega(g)$ is the modulus of continuity of $g \in C[a, b]$.

THEOREM 1.2 (Markoff inequality, [7, pp. 134–141]):

$$\| P_n^{(k)} \| \leqslant M n^{2k} \| P_n \|, \qquad n \geqslant 1$$

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and Bernstein's Inequality ([7], page 133)

$$\| P_n^{(k)} \|_{[\alpha,\beta]} \leqslant N n^k \| P_n \|, \qquad n \geqslant 1.$$

The constant M depends on k and the constant N depends on k, α, β.

THEOREM 1.3 [3, p. 39]. *There exists a constant K such that, if $f \in C'[a, b]$,*

$$E_n(f) \leqslant \frac{K}{n} E_{n-1}(f'), \qquad n \geqslant 1.$$

The behavior of the derivatives of the trigonometric polynomial of best approximation has been investigated by Czipszer and Freud [2] and by Zamansky [10]. We show here that, with proper restrictions, some of their results can be extended to the algebraic case. See also [4, 6].

II. CONVERGENCE OF THE SEQUENCE OF DERIVATIVES OF THE POLYNOMIAL OF BEST APPROXIMATION

In this section we study k's for which $\lim_{n\to\infty} \| P_n^{(k)} - f^{(k)} \|_{[c,d]} = 0$, $a \leqslant c < d \leqslant b$, as well as the corresponding speeds of convergence, where P_n is the polynomial of best approximation to $f \in C^N[a, b]$. The main results are Theorems 2.4 and 2.8.

THEOREM 2.1. *There exists a constant M with the following property: Let $f \in C[a, b]$ be such that, for some λ, $E_n(f) \leqslant \lambda/n$, $n \geqslant 1$, $E_0(f) \leqslant \lambda$. Then, for P_n, the polynomial of best approximation to f, one has:*

$$\| P_n' \| \leqslant M\lambda n, \qquad n \geqslant 1.$$

Proof. Let k be defined by $2^k \leqslant n < 2^{k+1}$. Then

$$P_n = P_n - P_{2^k} + \sum_{i=1}^{k} (P_{2^i} - P_{2^{i-1}}) + (P_1 - P_0) + P_0 .$$

By differentiating both sides of this identity and applying Markoff's inequality, we obtain:

$$\| P_n' \| \leqslant K(n^2 \| P_n - P_2 \| + \sum_{i=1}^{k} 2^{2i} \| P_{2^i} - P_{2^{i-1}} \| + \| P_1 - P_0 \|).$$

Now, for $l < m$,

$$\| P_m - P_l \| \leqslant \| P_m - f \| + \| P_l - f \| \leqslant E_m(f) + E_l(f) \leqslant 2E_l(f);$$

hence

$$\| P_n' \| \leqslant K(2n^2 E_{2^k}(f) + \sum_{i=1}^{k} 2^{2i-1} E_{2^{i-1}}(f) + 2E_0(f))$$

$$\leqslant K\left(2 \cdot 2^{2(k+1)} \frac{\lambda}{2^k} + \sum_{i=1}^{k} 2^{2i+1} \frac{\lambda}{2^{i-1}} + 2\lambda\right)$$

$$\leqslant K\lambda\left(42^k + 4 \sum_{i=1}^{k} 2^i + 2\right) \leqslant M\lambda n.$$

THEOREM 2.2. *There exists a constant M with the following property: if a function f satisfies $|f(x) - f(y)| \leqslant \lambda \,|\, x - y \,|$, $x, y \in [a, b]$, then for the polynomial P_n of best approximation to f,*

$$\| P_n' \| \leqslant M\lambda n, \qquad n \geqslant 1.$$

Proof. This is a direct consequence of Jackson's theorem and Theorem 2.1.

THEOREM 2.3. *Let k, N be integers with $0 \leqslant k \leqslant N$. There exists a constant M, depending on N, such that, if P_n is the polynomial of best approximation to $f \in C^N[a, b]$, then*

$$\| P_n^{(k)} - f^{(k)} \| \leqslant Mn^k E_{n-k}(f^{(k)}), \qquad n \geqslant k.$$

Proof. The theorem is true for $N = 0$. Let $N \geqslant 0$ and suppose that

$$\| P_n^{(k)} - h^{(k)} \| \leqslant M_N n^k E_{n-k}(h^{(k)}), \qquad 0 \leqslant k \leqslant N, n \geqslant k,$$

for every $h \in C^N[a, b]$, where P_n is the polynomial of best approximation to h. Let $f \in C^{N+1}[a, b]$. By the induction hypothesis, we have:

$$\| f^{(k+1)} - Q_{n-1}^{(k)} \| \leqslant M_N n^k E_{n-1-k}(f^{(k+1)}), \qquad 0 \leqslant k \leqslant N, n \geqslant k, \qquad (1)$$

where Q_{n-1} is the polynomial of best approximation to f'. Let $g(x) = f(x) - f(a) - \int_a^x Q_{n-1}(t)\, dt$, $x \in [a, b]$. Now, for $x, y \in [a, b]$, we have

$$| g(x) - g(y) | \leqslant \int_x^y | f'(t) - Q_{n-1}(t) |\, dt \leqslant E_{n-1}(f') | x - y | .$$

That is, g satisfies Lipschitz condition with constant $E_{n-1}(f')$. Let R_n be the polynomial of best approximation to g. We have, by Theorem 2.2:

$$\| R_n' \| \leqslant K_1 n E_{n-1}(f'), \qquad n \geqslant 1,$$

and, by Markoff's inequality and Theorem 1.3:

$$\begin{aligned}\| R_n^{(k)} \| &\leqslant K_k n n^{2(k-1)} E_{n-1}(f') \\ &\leqslant K_k' \frac{n^{2k-1}}{(n-1)(n-2)\cdots(n-(k-1))} E_{n-k}(f^{(k)}) \\ &\leqslant K_k'' n^k E_{n-k}(f^{(k)}), \; k \leqslant N+1, \qquad n \geqslant k. \end{aligned} \tag{2}$$

From (1) and (2) we conclude that

$$\begin{aligned}\| f^{(k)} - Q_n^{(k-1)} - R_n^{(k)} \| &\leqslant K_k'' n^k E_{n-k}(f^{(k)}) + M_N n^{k-1} E_{n-k}(f^{(k)}) \\ &\leqslant M_{N+1} n^k E_{n-k}(f^{(k)}), \; k \leqslant N+1, \, n \geqslant k.\end{aligned}$$

The theorem follows because $-f(a) + \int_a^x Q_{n-1}(t)\, dt + R_n(x)$ is the polynomial of best approximation to f.

THEOREM 2.4. *Let k, N be integers with $0 \leqslant k \leqslant N/2$. There exist constants S and T which depend on N such that, if $f \in C^N[a, b]$ and P_n is the polynomial of best approximation to f, then*

$$\| P_n^{(k)} - f^{(k)} \| \leqslant S E_{n-2k}(f^{(2k)}) \leqslant T \frac{1}{n^{N-2k}} E_{n-N}(f^{(N)}), \qquad n \geqslant N.$$

Proof. This is a direct consequence of Theorems 1.3 and 2.3.

COROLLARY 2.5. *Let k, N, f, P_n be as in Theorem 2.4. There exists a constant M, which depends on N, such that*

$$\| P_n^{(k)} - f^{(k)} \| \leqslant M \frac{1}{n^{N-2k}} \omega \left(f^{(N)}, \frac{1}{n} \right), \qquad n > N.$$

Proof. By Jackson's theorem and the properties of the modulus of continuity, we have, for $n > N$,

$$E_{n-N}(f^{(N)}) \leqslant K \omega \left(f^{(N)}, \frac{1}{n-N} \right) \leqslant K \left(\frac{1}{n-N} + 1 \right) \omega \left(f^{(N)}, \frac{1}{n} \right)$$

The corollary follows from Theorem 2.4.

Corollary 2.5 was obtained by Roulier [8]. We now show that Theorem 2.4 improves Corollary 2.5. We first need a preliminary result.

PROPOSITION 2.6. *Let $f(x) = (x+1)^{1/2}$, $x \in [-1, 1]$. Then $E_n(f) \leqslant K/n$, $n \geqslant 1$.*

Proof. By [7, p. 120], $E_n(f) = E_n^*(f(\cos x))$ where E_n^* is the degree of approximation by trigonometric polynomials of order at most n. Now $(\cos x + 1)^{1/2} = 2^{1/2} \mid \cos x/2 \mid$ has a derivative bounded by $2^{1/2}/2$ (except at the odd multiples of π where the derivative does not exist). It follows that $|(\cos x + 1)^{1/2} - (\cos y + 1)^{1/2}| \leqslant 2^{1/2}/2 \mid x - y \mid$, $x, y \in [-1, 1]$. Now, Jackson's theorem [7, p. 84] implies that $E_n^*(\cos x + 1)^{1/2}) \leqslant K_1/n$, $n \geqslant 1$, and so $E_n((x + 1)^{1/2}) \leqslant K/n$, $n \geqslant 1$.

Let $f(x) = (x - 1)^2(x - 1)^{1/2}$, $x \in [-1, 1]$, so that $f''(x) = (15/4)(x + 1)^{1/2}$. Corollary 2.5 implies that $\| P_n' - f' \| \leqslant M/n^{1/2}$, while Theorem 2.4 implies that $\| P_n' - f' \| \leqslant T/n$. Of course, there are functions for which Theorem 2.4 does not yield more information than Corollary 2.5, for instance $f(x) = x^2 \mid x \mid$, $x \in [-1, 1]$ [7, p. 171].

The remainder of this section is devoted to proving the analog of Theorem 2.4 where the norm of P_n is taken over a subinterval of $[a, b]$.

THEOREM 2.7. *Let $a < \alpha < \beta < b$. There exists a constant M, depending on α and β, with the following property: if for a function f,*

$$| f(x) - f(y)| \leqslant \lambda \mid x - y \mid, \qquad x, y \in [a, b],$$

then

$$\| P_n' \|_{[\alpha,\beta]} \leqslant M\lambda, \qquad n \geqslant 1,$$

where P_n is the polynomial of best approximation to f on $[a, b]$.

Proof. There exists a sequence of (Q_n) of polynomials such that [3, p. 125]

$$\| Q_n - f \| \leqslant \frac{N\lambda}{n} \text{ and } \| Q_n' \|_{[\alpha,\beta]} \leqslant M\lambda, \qquad n \geqslant 1,$$

where N depends on α and β.

Now

$$\| P_n' \|_{[\alpha,\beta]} \leqslant \| P_n' - Q_n' \|_{[\alpha,\beta]} + \| Q_n' \|_{[\alpha,\beta]}$$

and

$$\begin{aligned} \| P_n' - Q_n' \|_{[\alpha,\beta]} &\leqslant Kn \| P_n - Q_n \|_{[a,b]} \\ &\leqslant Kn \left(E_n(f) + \frac{N\lambda}{n}\right) \\ &\leqslant Kn \left(\frac{K_1\lambda}{n} + \frac{N\lambda}{n}\right), \ n \geqslant 1, \end{aligned}$$

by Jackson's theorem and Bernstein's inequality. The theorem follows.

THEOREM 2.8. *Let $a < \alpha < \beta < b$ and let N and k be integers with*

$0 \leqslant k \leqslant N$. *There exists a constant* M, *which depends on* N, α, β *such that, if* P_n *is the polynomial of best approximation to* $f \in C^N[a, b]$, *then*

$$\| P_n^{(k)} - f^{(k)} \|_{[\alpha,\beta]} \leqslant M E_{n-k}(f^{(k)}), \qquad n \geqslant k.$$

The proof of this theorem is similar to that of Theorem 2.3, but requires a more careful use of Bernstein's inequality. Theorem 2.8 is true for $N = 0$. Suppose that

$$\| P_n^{(k)} - h^{(k)} \|_{[\alpha,\beta]} \leqslant M_N E_{n-k}(h^{(k)}),$$

for every $h \in C^N[a, b]$, $0 \leqslant k \leqslant N$, $n \geqslant k$.

Let $f \in C^{N+1}[a, b]$. By the induction hypothesis we have:

$$\| f^{(k+1)} - Q_{n-1}^{(k)} \|_{[\alpha,\beta]} \leqslant M_N E_{n-1-k}(f^{(k+1)}), \qquad 0 \leqslant k \leqslant N, n \geqslant k, \tag{3}$$

where Q_{n-1} is the polynomial of best approximation to f' on $[a, b]$.

Define g as in Theorem 2.3. Then g satisfies Lipschitz condition with constant $E_{n-1}(f')$. Let R_n be the polynomial of best approximation to g on $[a, b]$. We have, by Theorem 2.7,

$$\| R_n' \|_{[c,d]} \leqslant K_1 E_{n-1}(f'), \qquad n \geqslant 1,$$

where $c = (a + \alpha)/2$, $d = (\beta + b)/2$.

By Bernstein's inequality and Theorem 1.3, we have:

$$\begin{aligned} \| R_n^{(k)} \|_{[\alpha,\beta]} &\leqslant K_k n^{k-1} \| R_n' \|_{[c,d]} \\ &\leqslant K_k K_1 n^{k-1} E_{n-1}(f') \\ &\leqslant K_k' \frac{n^{k-1}}{(n-1)(n-2)\cdots(n-(k-1))} E_{n-k}(f^{(k)}) \\ &\leqslant K_k'' E_{n-k}(f^{(k)}), \qquad k \leqslant N+1, n \geqslant k. \quad (4) \end{aligned}$$

From (3) and (4) we conclude that

$$\begin{aligned} \| f^{(k)} - P_n^{(k)} \|_{[\alpha,\beta]} &= \| f^{(k)} - Q_n^{(k-1)} - R_n^{(k)} \|_{[\alpha,\beta]} \\ &\leqslant M_{N+1} E_{n-k}(f^{(k)}), \qquad k \leqslant N+1, n \geqslant k. \end{aligned}$$

III. Divergence of the Sequence of Derivatives of the Polynomial of Best Approximation

Let P_n be the polynomial of best approximation to $f \in C^N[a, b]$. We now investigate the behavior, as $n \to \infty$, of $\| P_n^{(k)} \|_{[c,d]}$, $a < c < d < b$ for the k's which have not been considered in Section II.

THEOREM 3.1. *Let k, N be integers with $[N/2]+1 \leqslant k \leqslant N$. Let P_n be the polynomial of best approximation to $f \in C^N[a,b]$. Then there exist constants M, M_1, M_2 which depend on N, such that*

$$\begin{aligned} \| P_n^{(k)} \| &\leqslant \| f^{(k)} \| + M_1 n^k E_{n-k}(f^{(k)}) \\ &\leqslant \| f^{(k)} \| + M_2 n^{2k-N} E_{n-N}(f^{(N)}) \\ &\leqslant \| f^{(k)} \| + M n^{2k-N} \omega\left(f^{(N)}, \frac{1}{n}\right), \qquad n \geqslant N. \end{aligned}$$

Proof. This is a direct consequence of Theorems 2.3, 1.3 and the properties of the modulus of continuity.

THEOREM 3.2. *Let k, N be integers with $k > N \geqslant 0$. Let $f \in C^N[a,b]$, f not a polynomial. Then there exists a constant M, which depends on k and f, such that, if P_n is the polynomial of best approximation to f,*

$$\| P_n^{(k)} \| \leqslant M n^{2k-N} \omega\left(f^{(N)}, \frac{1}{n}\right), \qquad n \geqslant 1.$$

We need two preliminary remarks: First [9, p. 100], if $f \in C[a,b]$ is not a constant, we have $\omega(f, 1/n) \geqslant C/n$, $C > 0$, $n \geqslant 1$. Second, let $f \in C^N[a,b]$, $c < a$, $d > b$. We can extend f to $g \in C^N[c,d]$ in such a way that $\omega(g^{(N)}, h) \leqslant l\omega(f^{(N)}, h)$, $h > 0$, l being a constant depending on f. Indeed, let $g(x) = \sum_{n=0}^{N} (f^{(n)}(a)/n!)(x-a)^n$ if $c \leqslant x < a$, $g(x) = f(x)$ if $a \leqslant x \leqslant b$, and $g(x) = \sum_{n=0}^{N} (f^{(N)}(b)/n!)(x-b)^n$ if $b < x \leqslant d$. Then g is as required.

Proof of Theorem 3.2. Let c, d and g be as above. By Theorem 2.8, there exists a sequence of polynomials Q_n and a constant K such that

$$\begin{aligned} \| Q_n^{(k)} - g^{(k)} \|_{[c,d]} &\leqslant \frac{K}{n^{N-k}}\, \omega\left(g^{(N)}, \frac{1}{n}\right), \\ &\leqslant \frac{Kl}{n^{N-k}}\, \omega\left(f^{(N)}, \frac{1}{n}\right), \qquad k \leqslant N, \quad n \geqslant k+1. \end{aligned}$$

It follows that $\| Q_n^{(k)} \|_{[c,d]} \leqslant K_k'$, $0 \leqslant k \leqslant N$, and $\| Q_n^{(k)} \|_{[a,b]} \leqslant K_k'' n^{k-N}$, $k > N$, by Bernstein's inequality. So

$$\| Q_n^{(k)} \|_{[a,b]} \leqslant \max(K_k', K_k'' n^{k-N}), \qquad k \geqslant 0.$$

Now we have, for $k \geqslant 0$:

$$\| P_n^{(k)} \|_{[a,b]} \leqslant \| P_n^{(k)} - Q_n^{(k)} \|_{[a,b]} + \| Q_n^{(k)} \|_{[a,b]}\,.$$

Also

$$\| P_n^{(k)} - Q_n^{(k)} \|_{[a,b]} \leqslant S_k n^{2k} \| P_n - Q_n \|_{[a,b]}$$

$$\leqslant N_k n^{2k} \left(E_n(f) + \frac{Kl}{n^N} \omega \left(f^{(N)}, \frac{1}{n}\right)\right).$$

So Jackson's theorem yields:

$$\| P_n^{(k)} \|_{[a,b]} \leqslant K_k l n^{2k-N} \omega \left(f^{(N)}, \frac{1}{n}\right) + \max(K_k', K_k'' n^{k-n}).$$

But $\omega(f^{(N)}, 1/n) \geqslant C/n$ because $f^{(N)}$ is not a constant. It follows that

$$\| P_n^{(k)} \|_{[a,b]} \leqslant K_k l n^{2k-N} \omega \left(f^{(N)}, \frac{1}{n}\right) + C_n \omega \left(f^{(N)}, \frac{1}{n}\right) \max(K_k', K_k'' n^{k-N})$$

$$\leqslant \omega \left(f^{(N)}, \frac{1}{n}\right) (K_k l 2^{2k-N} + C_n \max(K_k', K_k'' n^{k-N})).$$

But if $k \geqslant N+1$, then $2k - N \geqslant k - N + 1$. It follows that $\| P_n{}^k \|_{[a,b]} \leqslant Mn^{2k-N} \omega(f^{(N)}, 1/n)$ for $k \geqslant N+1$.

THEOREM 3.3. *Let k, N be integers with $k > N \geqslant 0$. Let $a < \alpha < \beta < b$, $0 < \epsilon \leqslant 1$ and $K > 0$. Let $| f^{(N)}(x) - f^{(N)}(y)| \leqslant K | x - y |^\epsilon$, $x, y \in [a, b]$. There exists a constant M which depends on α, β, K, k, N such that, if P_n is the polynomial of best approximation to f on $[a, b]$,*

$$\| P_n^{(k)} \|_{[\alpha,\beta]} \leqslant Mn^{k-N-\epsilon}, \qquad n \geqslant 1.$$

We first exclude the possibility $k = N + \epsilon$. Let l be defined by $2^l \leqslant n < 2^{l+1}$.

Then

$$P_n = P_n - P_{2^l} + \sum_{i=1}^{l} (P_{2^i} - P_{2^{i-1}}) + (P_1 - P_0) + P_1 .$$

By Bernstein's inequality and Jackson's theorem, we obtain, as in the proof of Theorem 2.1:

$$\| P_n^{(k)} \|_{[\alpha,\beta]} \leqslant R_k \left(n^k \| P_n - P_{2^l} \| + \sum_{i=1}^{l} {}^{ki} \| P_{2^i} - P_{2^{i-1}} \| + \| P_i - P_0 \|\right)$$

and

$$\| P_n^{(k)} \|_{[\alpha,\beta]} \leqslant R_k' \left(n^k K' n^{-N-\epsilon} + K'' \sum_{i=1}^{l} 2^{ki} 2^{-(i-1)(N+\epsilon)} + E_1(f) + E_0(f)\right).$$

The theorem is proved for $k \neq N + \epsilon$.

We now prove the theorem for $k = N + 1$ and for $a = -1$, $b = 1$. The general case is reduced to this by the transformation $x \mapsto \frac{1}{2}(n - a)x + \frac{1}{2}(b + a)$. Let $g(x) = f(\cos x)$, $x \in [-\pi, \pi]$. Then

$$| g^{(N)}(x) - g^{(N)}(y)| \leqslant K' \mid x - y \mid, \qquad x, y \in (-\pi, \pi).$$

Now, there exists a sequence of even positive kernels K_n such that $\int_{-\Pi}^{\Pi} K_n(t)\, dt = 1$,

$$T_n(x) = - \int_{-\pi}^{\pi} K_n(t) \sum_{i=1}^{k} (-1)^i \binom{k}{i} g(x + it)\, dt$$

is a trigonometric polynomial of degree n, and

$$\| g - T_n \| \leqslant M'n^{-k} \qquad [5, \text{p. } 57].$$

It follows that

$$\| T_n^{(j)} \| \leqslant \sum_{i=1}^{k} \binom{k}{i} \int_{-\pi}^{\pi} | g^{(j)}(x + it)|\, dt \leqslant K_j\,, \qquad 1 \leqslant j \leqslant k.$$

(The case $j = k$ follows from the fact that $g^{(k)}$ exists almost everywhere and is bounded.) Let $Q_n(x) = T_n(\arccos x)$, $x \in [-1, 1]$. Because g and K_n are even, T_n is even and so Q_n is an algebraic polynomial. Now

$$Q_n^{(k)}(x) = \sum_{i=1}^{k} T_n^{(i)}(\arccos x)\, V_i(x)\, W_i((1 - x^2)^{-1/2}),$$

where V_i and W_i are algebraic polynomials of degree bounded by $k - 1$ and $2k - 1$ respectively. It follows that, if $-1 < \alpha < \beta < 1$, then

$$\| Q_n^{(k)} \|_{[\alpha,\beta]} \leqslant K''.$$

Also

$$\| P_n^{(k)} \|_{[\alpha,\beta]} \leqslant \| P_n^{(k)} - Q_n^{(k)} \|_{[\alpha,\beta]} + \| Q_n^{(k)} \|_{[\alpha,\beta]}$$

and

$$\begin{aligned} \| P_n^{(k)} - Q_n^{(k)} \|_{[\alpha,\beta]} &\leqslant Kn^k \| P_n - Q_n \|_{[-1,1]} \\ &\leqslant Kn^k(E_n(f) + Mn^{-k}) \\ &\leqslant Kn^k(K_2 n^{-k} + K_1 n^{-k}) \end{aligned}$$

by Jackson's theorem and Bernstein's inequality. The proof of the theorem is complete.

IV. Remarks and Open Questions

We can somewhat generalize Theorems 2.4 and 2.8: If $f \in C^N[a, b]$ and $\| P_n - f \| = O(E_n(f))$, then $\| P_n^{(k)} - f^{(k)} \| = O(E_{n-2k}(f^{2k}))$, $0 \leqslant k \leqslant N/2$, and $\| P_n^{(k)} - f^{(k)} \|_{[\alpha,\beta]} = O(E_{n-k}(f^{(k)}))$, $0 \leqslant k \leqslant N$, $a < \alpha < \beta < b$. Theorem 2.8 extends to the trigonometric case. Let C^N be the space of everywhere N-times continuously differentiable functions of period 2π. If $\| T_n - f \|_{[-\pi,\pi]} = O(E_n^*(f))$, then $\| T_n^{(k)} - f^{(k)} \|_{[-\pi,\pi]} = O(E_n^*(f^{(k)}))$, $0 \leqslant k \leqslant N$, where T_n is a trigonometric polynomial of degree at most n and $E_n^*(f)$ is the degree of approximation by such polynomials.

Indeed, it suffices to notice that Theorem 2.7 holds true for the trigonometric case, to use the corresponding Bernstein's inequality [7, p. 90], $\| T_n' \| \leqslant n \| T_n \|$, and to observe that $E_n^*(f) \leqslant (K/n)\, E_n^*(f')$ if $f \in C'[-\pi, \pi]$. The last result was found by Czipszer and Freud [2]. Similarly, by using the above quoted inequality in the proof of Theorem 3.4, we obtain that if $f \in C^N[-\pi, \pi]$, $k > N > 0$, $0 < \epsilon \leqslant 1$ and $| f^{(N)}(x) - f^{(N)}(y)| \leqslant K | x - y |^\epsilon$, $x, y \in [-\pi, \pi]$, then there is a constant M which depends on k and f such that, if $\| T_n - f \| = E_n^*(f)$, then $\| T_n^{(k)} \| \leqslant M m^{k-N+\epsilon}$, $n \geqslant 1$. For related results see [4].

Let $f \in C^N[-\pi, \pi]$, $k > N \geqslant 0$. We conjecture that there is no constant M which depends only on k and f such that if $\| T_n - f \| = E_n^*(f)$, then $\| T_n^{(k)} \| \leqslant M n^{k-N} \omega(f^{(N)}, 1/n)$, $n \geqslant 1$. Similarly for the algebraic case.

We make also the following conjecture: for every $N > 1$ there exists $f \in C^{2N-1}[a, b]$ such that, for all k, $N \leqslant k \leqslant 2N - 1$, $P^{(k)}(a)$ does not converge to $f^{(k)}(a)$, where P_n is the polynomial of best approximation to f.

It is interesting to notice that we cannot replace the hypothesis of Theorem 2.7 by those of Theorem 2.1. Indeed, we have

Theorem 4.1. *Let $a < \alpha < \beta < b$, and let $\lambda > 0$. There exists a constant M which depends on α, β, λ, with the following property: let $f \in C[a, b]$ satisfy $E_n(f) \leqslant \lambda/n$, $n \geqslant 1$; $E_0(f) \leqslant \lambda$. Then, for the polynomial P_n of best approximation to f, one has:*

$$\| P_n' \|_{[\alpha,\beta]} \leqslant M \log n, \qquad n \geqslant 2.$$

The proof is almost exactly the same as the first part of the proof of Theorem 3.4.

The next theorem illustrates Theorem 4.1.

Theorem 4.2. *There exists a function $f \in C[-1, 1]$ such that $E_n(f) \leqslant K/n$ and, if P_n is the polynomial of best approximation to f on $[-1, 1]$, $\| P_n' \|_{[\alpha,\beta]} \geqslant K \log n$, $n = 1, 2, \ldots$, whenever $-1 < \alpha < \beta < 1$.*

Proof. Let $f(x) = \sum_{k=0}^{\infty} 5^{-k} T_{5^k}(x)$, where $T_n(x) = \cos(n \arccos x)$. For this function we have [11] $E_n(f) \leqslant K/n$. On the other hand,

$$P_{5^n}(x) = \sum_{k=0}^{n} 5^{-k} T_{5^k}(x) \tag{4.1}$$

is the polynomial of degree at most 5^n of best approximation to f (see [7, p. 127]). Since

$$P'_{5^n}(x) = \sum_{k=0}^{n} \sin(5^k \arccos x) \frac{1}{(1 - x^2)^{1/2}}$$

and $5^k \equiv 1 \pmod 4$,

$$P'_{5^n}(0) = \sum_{k=0}^{n} 1 = (n + 1).$$

As $P_{5^n}(x)$ is the polynomial of degree $\leqslant k$ of best approximation to f, for $k = 5^n, 5^n + 1, \ldots, 5^{n+1} - 1$ [1, p. 127], the theorem follows.

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